

# PLASTIC HINGE LOCATIONS IN STEEL COLUMNS

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## SUMMARY

Some standards for the design of steel structures attempt to encourage column yielding at the member ends, rather than along the column length during inelastic action such as may occur during strong earthquake shaking. In this paper, effects of residual stress and stability are considered separately and then combined to obtain equations that are more simple and more accurate than those in current standards.

## 1. INTRODUCTION

Australian and New Zealand steel design standards [6, 7] require that plastic hinges in yielding columns of steel frames should occur at the member ends, rather than along the member length as shown in Figure 1. Such a requirement is not present in the standards of other countries.



(a) Hinge at member end (b) Hinge along member length

**Figure 1: Moment Patterns for Different Hinge Positions (based on Lay [2]).**

Flexural yielding along the length of a column during seismic response is considered to be undesirable because (a) it is difficult to effectively brace along the length to restrict local and lateral buckling, (b) the rotational capacity is likely to be less than that from columns yielding at the member ends where there may be some type of bracing restraint, (c) cumulative hinge rotations in one direction may occur during cyclic loading, and (d) the correct collapse mechanism and hinge rotational demands are hard to predict. Also, most testing is conducted such that yielding occurs at the column ends, so there is little information about yielding along a column length.

The Australian steel structures standard, AS4100 [6], specifies that the seismic design axial compression force,  $N^*$ , for columns shall satisfy Equations 1 and 2, where  $\phi$  is the strength reduction factor,  $\beta$  is the end moment ratio which is positive in double curvature,  $\lambda$  is the member in-plane slenderness factor given by Equation 3,  $N_{OL}$  is the Euler in-plane buckling force given in Equation 4,  $I$  is the member second moment of area for bending in the direction considered,  $E$  is the elastic modulus,  $L$  is the member length,

$N_S$  is the nominal section capacity given by Equation 5,  $A$  is the nominal cross-sectional area, and  $\sigma_Y$  is the material nominal yield stress. These equations were developed to discourage yielding along the member length and they controlled column sizes in some braced frames. A plot of the permitted  $N^*/(\phi N_S)$ , for a particular end moment ratio,  $\beta$ , and slenderness limit,  $\lambda$ , is given in Figure 2. It can be seen that there is a discontinuity at the axial force ratio of 0.15.

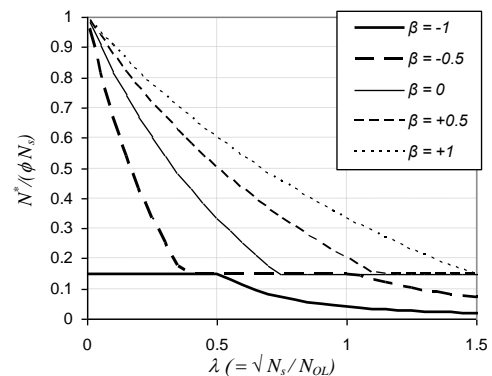
$$\frac{N^*}{\phi N_S} \leq \left[ \frac{1 + \beta - \lambda}{1 + \beta + \lambda} \right] \quad \text{when} \quad \frac{N^*}{\phi N_S} > 0.15 \quad (1)$$

$$\frac{N^*}{\phi N_S} \leq \left[ \frac{0.6 + 0.4\beta}{\lambda} \right]^2 \quad \text{when} \quad \frac{N^*}{\phi N_S} \leq 0.15 \quad (2)$$

$$\lambda = \sqrt{\frac{N_S}{N_{OL}}} \quad (3)$$

$$N_{OL} = \frac{\pi^2 EI}{L^2} \quad (4)$$

$$N_S = A\sigma_Y \quad (5)$$



**Figure 2: AS4100 Axial Force Limits for Different Member Slenderness and End Moment Ratio.**

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For axial force ratios,  $N^*/(\phi N_s)$ , greater than 0.15, the source book [2] which provides background information to the provisions in the NZ and Australian steel structures standard, states that Equations 1 and 2 were developed by curve fitting column deflection curve data in Lay's PhD thesis [1] for highly axially loaded members, in conjunction with an analytical approach to keep the hinges away from the member ends and to guarantee rotation capacity. However, the degree of rotation capacity to be provided by these equations is not described. Lay's thesis [1] does present this equation for the case of  $\beta = 0$ , but the exact means by which the equation was developed is not clear.

Before the 2007 amendment, the NZ steel structures standard provision (NZS3404:1997) was identical to that in the Australian standard. The NZ standard was amended based on studies by Peng [4, 5]. Peng reduced the elastic stiffness

ratio of the column to consider residual stress effects based on the column buckling curve, and used stability functions to determine the maximum axial force that could be applied to a member with a particular axial moment ratio,  $\beta$ , and slenderness ratio,  $\lambda$ , before the location of maximum moment moved away from the end of the column. This gave an exact numerical relationship for each of the 5 column residual stress categories, represented by the parameter  $\alpha_b$  in the Australian and NZ standards. To develop revised relationships for the NZ standard, empirical equations were fit to the data as shown in Figure 3. It may be seen that the actual curves, found from analysis results, are generally much less conservative than the AS4100/NZS3404:1997 curve (Lay's equations) for high axial force ratios. The equations that Peng proposed were also less conservative than Lay's equations but they were more conservative than that of the actual curves and especially for high axial force levels.

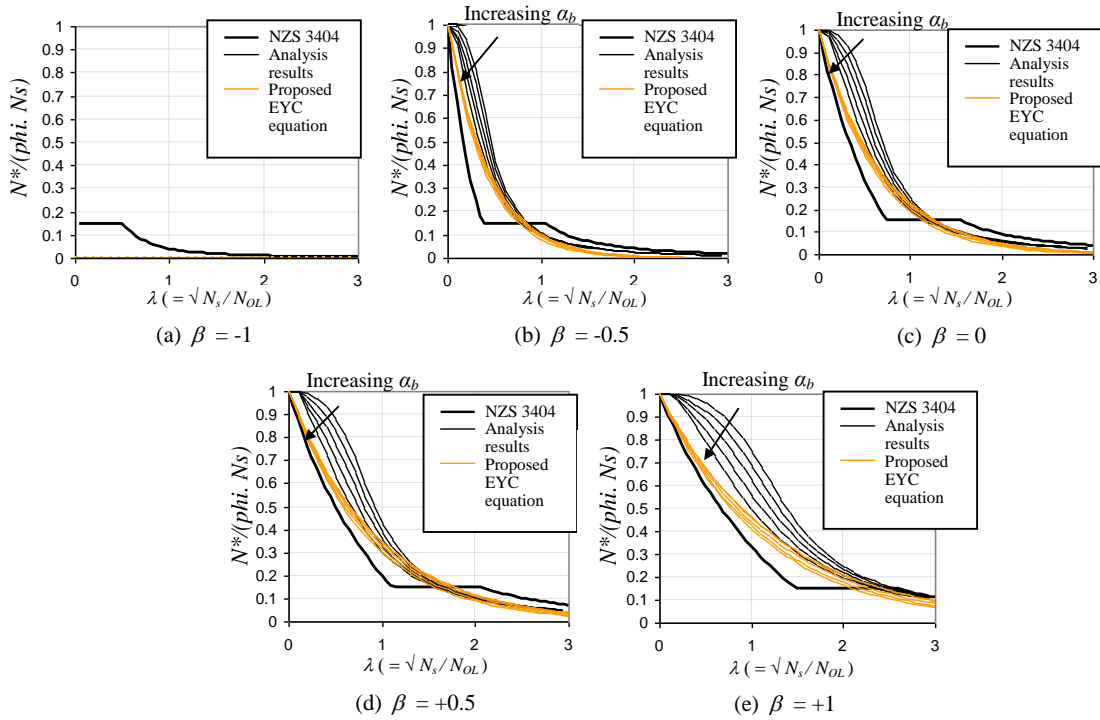


Figure 3: Comparison of AS 1250, Analysis Results and Peng's Proposed Equation for Different  $\beta$ .

Peng's proposed empirical equations are computed using the method below. They form the basis of the expression in the 2007 amendment to the NZ steel structures standard. The exponential function given in Equation 6 is used where three constants,  $A$ ,  $B$  and  $C$  vary with section constant,  $\alpha_b$ . The values of these constants for different values of  $\alpha_b$  are given in Table 1.

$$\frac{N^*}{\phi N_s} \leq \left\{ \frac{A \times (\beta_m + 1)^B}{e^{(C/\beta_m + 1)}} \right\}^\lambda \quad (6)$$

Table 1. Coefficients for Different Section Types

$\alpha_b$	$A$	$B$	$C$
1	0.235	0.95	0.21
0.5	0.247	0.91	0.19
0	0.263	0.88	0.19
-0.5	0.265	0.92	0.17
-1	0.276	0.87	0.19

The study described in this paper was conducted to develop a more accurate expression for the EYC equation by developing

expressions for residual stress effects and for stability effects separately, then combining them. The impetus for this work was a study for the direct analysis of steel frames subject to non-earthquake loadings where expressions for the reduction in flexural stiffness due to residual stresses were required [3]. It made sense to use the same expressions for residual stress both for direct analysis of steel frames, and for the end yielding criteria for steel columns.

## 2. RESIDUAL STRESS EFFECTS

### 2.1 Residual Stress Calculation Method

The value of  $EI_{eff}$  used above is computed relatively easily from the column design curves. For a specific axial force ratio, a point on a column design strength curve, such as point B in Figure 4, may be regarded as a point on an elastic Euler buckling curve with a reduced flexural stiffness,  $EI_{eff}$ . Since the buckling curves at points A and B have the same strength,  $N^*$ , and the corresponding effective lengths are  $kL$  and  $kL_{eff}$  respectively, the reduced flexural stiffness,  $EI_{eff}$ , may be found from Equations 7 and 8. Here  $SRF$  is a stiffness reduction factor relating the effective flexural stiffness,  $EI_{eff}$ , and the

elastic flexural stiffness,  $EI$ , ignoring residual stresses and out-of-straightness effects.

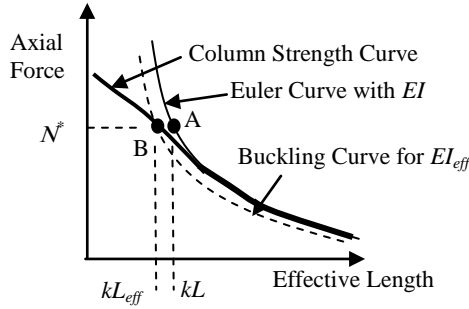


Figure 4: Computing Effective Stiffness,  $EI_{eff}$  (1).

$$N^* = \frac{\pi^2 (EI)_{eff}}{(kL)_{eff}^2} = \frac{\pi^2 EI}{(kL)^2} \quad (7)$$

$$(EI)_{eff} = (kL_{eff}/kL)^2 \cdot (EI) = SRF \cdot (EI) \quad (8)$$

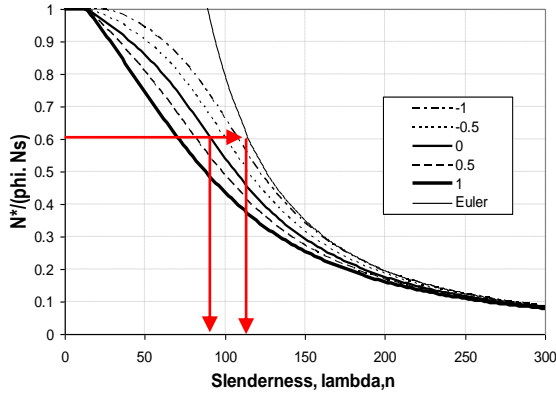


Figure 5: Computing Effective Stiffness,  $EI_{eff}$  (2).

The same approach is valid when other parameters, such as  $\lambda_{cr}$ , from the Australian and NZ standards, are used. For example, for an axial force ratio of 0.60 say,  $(EI)_{eff} = (91.2/114.5)^2 \cdot EI = 0.6351EI$  for  $\alpha_b = 0$  as shown in Figure 5.

## 2.2 Residual Stress Effect and Approximation

Stiffness reduction factors for NZS3404 column curves are given in Figure 6. As expected, when the axial force ratio is zero, there is no reduction of flexural stiffness and when the axial force ratio is unity, the flexural stiffness is zero. Also sections with greater residual stress effects, that is those with greater  $\alpha_b$ , have a lower SRF.

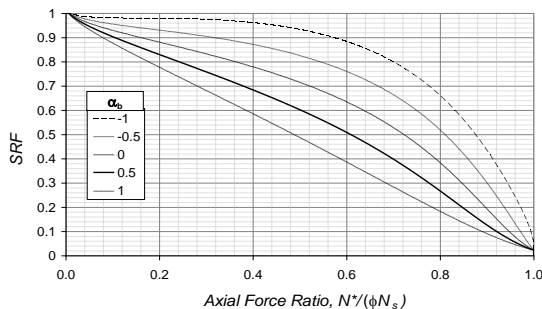


Figure 6: Stiffness Reduction Factor for NZS3404 Curves.

The curves may be approximated by an equation of the form in Equation 9 where the values of  $c$  for the five residual categories are given in Table 2.

$$SRF = 1 - \frac{x}{1 + c(1 - x)} \quad (9)$$

Table 2. Best-fit values for the coefficient  $c$

$\alpha_b$	-1	-0.5	0	0.5	1
$c$	8.9	2.9	0.9	0.2	-0.1

It is more convenient if equations, rather than an equation and a table as shown above, are used for design. An equation was developed by plotting the parameter  $c$  against the residual stress parameter  $\alpha_b$  and fitting an equation. The final expressions for SRF are given in Equations 10 and 11. A comparison of the actual SRF,  $SRF_{Actual}$ , with that obtained from these equations,  $SRF_{Eq10}$ , is given in Table 3 over the axial force ratio range of 0.0 to 1.0 and also in Figure 7. Since the maximum positive value indicates that  $SRF_{Actual}$  is greater than  $SRF_{Eq10}$ , Table 3 shows that  $SRF_{Eq10}$  is generally conservative. It can be seen from the figure that the approximation is non-conservative mainly at high levels of axial force which are not permitted in design.

$$SRF = 1 - \frac{\frac{N^*}{\phi N_s}}{\left(1 + c \left(1 - \frac{N^*}{\phi N_s}\right)\right)} \quad (10)$$

$$c = 1.5 \exp(-1.8\alpha_b) - 0.35 \quad (11)$$

Table 3: Equations 10 and 11 Compared to Actual SRF

	$SRF_{Actual} - SRF_{Eq10}$				
$\alpha_b$	-1	-0.5	0	0.5	1
max	0.050	0.023	0.050	0.053	0.023
min	-0.085	-0.027	-0.021	-0.015	-0.013

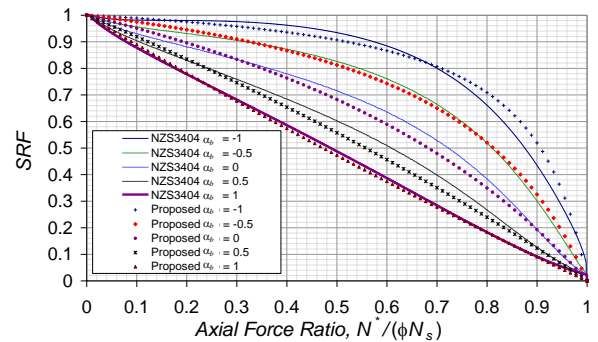


Figure 7: Comparison of the Actual SRF Obtained from the NZS3404 Column Curves with Equation 10 (Proposed) SRF.

## 3. STABILITY EFFECTS

Rather than develop the stability approach numerically, as per Peng *et al.* (2007), a closed form solution is proposed based on work by Lay (1964). It considers the combination of axial force, slenderness ratio,  $\lambda$ , and moment ratio,  $\beta$ , that would cause hinging away from the member end. Here, for the beam-column member in Figure 8 subjected to an axial compression force,  $N^*$ , and end moments,  $M^*$ , the horizontal deflection,

$v(z)$ , is given in Equations 12 where  $\theta$  is the stability coefficient in Equation 13.

$$v = \frac{M^*}{N^*} \left[ \frac{\beta + \cos \theta}{\sin \theta} \sin \left( z \frac{\theta}{L} \right) + 1 - \cos \left( z \frac{\theta}{L} \right) - (1 + \beta) \frac{z}{L} \right] \quad (12)$$

$$\theta = \sqrt{\frac{N^* L^2}{EI}} \quad (13)$$

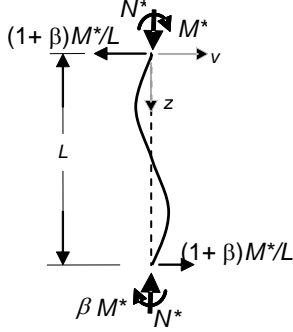


Figure 8: Deflected shape of beam-column member.

By differentiating Equation 12 three times with respect to  $z$  and by setting  $d^3v/dz^3$  equal to zero, the location of maximum moment,  $z_{max}$ , is found, as shown in Equation 14.

$$\tan \left( z_{max} \frac{\theta}{L} \right) = \frac{-\beta - \cos \theta}{\sin \theta} \quad (14)$$

To ensure the maximum moments occur only at the end of column, the location of maximum moment,  $z_{max}$ , must be zero. Therefore, substituting  $z_{max} = 0$  into Equation 14 gives Equation 15 which is the condition that must be satisfied if yielding is to be at the end of the member. This equation was also verified numerically (Lu, 2008).

$$\cos(\theta) + \beta \leq 0 \quad (15)$$

#### 4. END YIELDING CURVE DEVELOPMENT

To ensure that yielding will not occur away from the end of a member with a specified effective flexural stiffness,  $EI_{eff}$ , length,  $L$ , end moment ratio,  $\beta$ , subject to a factored axial force demand,  $N^*$ , Equation 16 must be satisfied. Here  $N^*_{max}$  is the maximum permitted axial compression force to prevent end yielding.

$$N^* \leq N^*_{max} \quad (16)$$

The following checks may be used:

##### a) Design method

This method may be used when the axial force on the column,  $N^*$ , is already known. It does not require iteration.  $N^*_{max}$  does not need to be computed explicitly. Instead, as long as  $N^* < N^*_{max}(N^*)$  then yielding will occur at the member ends, where  $N^*_{max}(N^*)$  is an initial estimate of  $N^*_{max}$  based on the applied axial force. If  $N^* = N^*_{max}(N^*)$ , then both of these values are also equal to  $N^*_{max}$ . The steps are:

- i) Compute the stiffness reduction factor,  $SRF$ , for the applied axial force ratio,  $N^*/\phi N_s$ , and column type,  $\alpha_b$ , from Figure 6, from tabulated values (e.g. Lu (2008), or use the Equation 10 approximation (which is shown again for convenience).

$$SRF = 1 - \frac{\frac{N^*}{\phi N_s}}{\left( 1 + c \left( 1 - \frac{N^*}{\phi N_s} \right) \right)} \quad (10)$$

$$c = 1.5 \exp(-1.8\alpha_b) - 0.35 \quad (11)$$

- ii) Compute the reduced stiffness,  $(EI)_{eff}$  :

$$(EI)_{eff} = SRF * EI \quad (17)$$

- iii) Compute  $\theta$  from Equation 18. Here  $\theta$  ranges from 0.0 when  $\beta = -1$  (single curvature), to  $\pi$  when  $\beta = 1$  (double curvature).

$$\theta = \cos^{-1}(-\beta) \quad (18)$$

- iv) Instead of the unreduced stiffness,  $EI$ , the reduced stiffness,  $EI_{eff}$ , is used to compute the maximum permitted axial force with Equation 19. When  $\beta = +1$  (double curvature),  $N^*_{max}$  is the Euler buckling load considering the effective flexural stiffness,  $EI_{eff}$ , and when  $\beta = -1$  (single curvature),  $N^*_{max}$  is zero implying that any axial force would cause the moment to move away from the ends as would be expected.

$$N^*_{max}(N^*) = \frac{\theta^2 (EI)_{eff}}{L^2} \quad (19)$$

- b) Direct method using equations

$N^*_{max}$  of any section can be calculated directly by using Equations 20 and 21 which were found by substituting Equation 10 into Equation 19 and equating  $N^*_{max}$  and  $N^*$ . Here  $c$  is from Equation 11.

$$N^*_{max} = \phi N_s \left[ \frac{(1 + \omega)(c + 1) - \sqrt{((1 + \omega)(c + 1))^2 - 4c\omega(1 + c)}}{2c} \right] \quad (20)$$

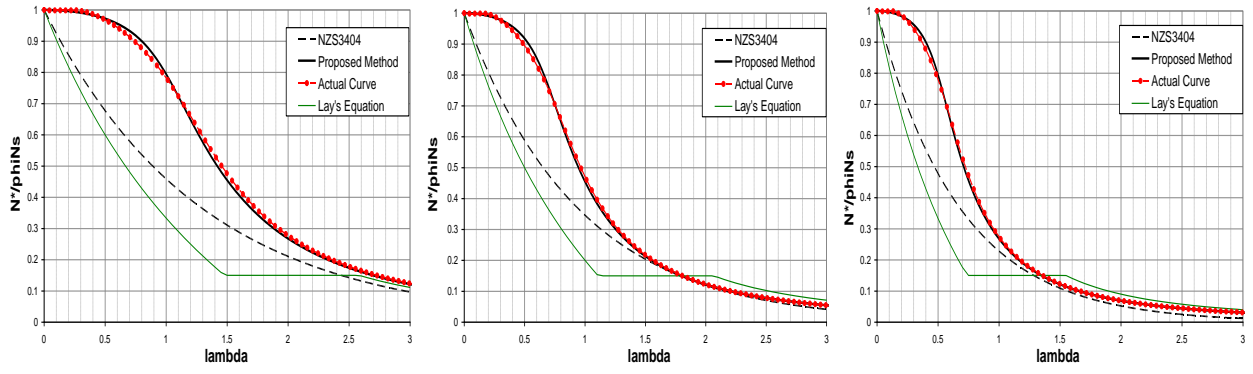
$$\omega = \frac{1}{\phi} \left( \frac{\theta}{\pi \lambda} \right)^2 \quad (21)$$

For example, consider a 250UC89.4 column with  $E = 200 \times 10^9$  Pa,  $I = 143 \times 10^{-6}$  m<sup>4</sup>,  $A = 11,400$  mm<sup>2</sup>,  $N_s = 3420$  kN,  $\sigma_y = 300$  MPa,  $\phi = 0.9$  and  $\alpha_b = 0$ . The column length,  $L$ , is 3.163 m and it is subjected to single curvature bending with  $\beta = -0.5$ . Hence,  $c = 1.15$ ,  $\theta = \cos^{-1}(-\beta) = 1.047$ ,  $\lambda$  is 0.348 from Equation 3 and  $\omega = 1.018$ . Equation 20, from the direct method, gives  $N^*_{max} = 1,847$  kN and  $N^*_{max}/(\phi N_s) = 0.60$ . By taking this  $N^*_{max}$  as the axial force applied to the column then  $SRF = 0.59$  from Equation 10 and  $N^*_{max} = 1,847$  kN from Equation 19 indicating that the *design method* gives the same results as the *direct method*. For the axial force ratio of  $N^*_{max}/(\phi N_s) = 0.60$ , the actual  $SRF$  found from Figure 6 without the Equation 10 approximation is 0.635. This is slightly greater than the Equation 10 value of 0.60 and it results in a slightly greater value of  $N^*_{max}$  in this case. It should be noted that the actual curve from Peng's paper was developed with the safety reduction factor,  $\phi$ , was 1. However, in this paper, the curves were developed with  $\phi = 0.9$  since the results are dependent on  $\phi$ .

- c) Direct method using graphs

The same example is shown in Figure 10d where  $N^*_{max}/(\phi N_s)$  of 0.60 is obtained directly using  $\lambda = 0.348$ .

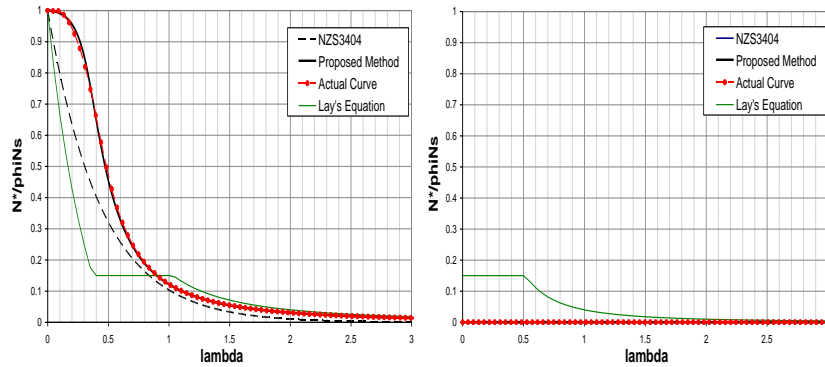
If a column does not satisfy the end yielding criteria of Equation 16 then a larger column will be required.



(a)  $\beta = 1$

(b)  $\beta = 0.5$

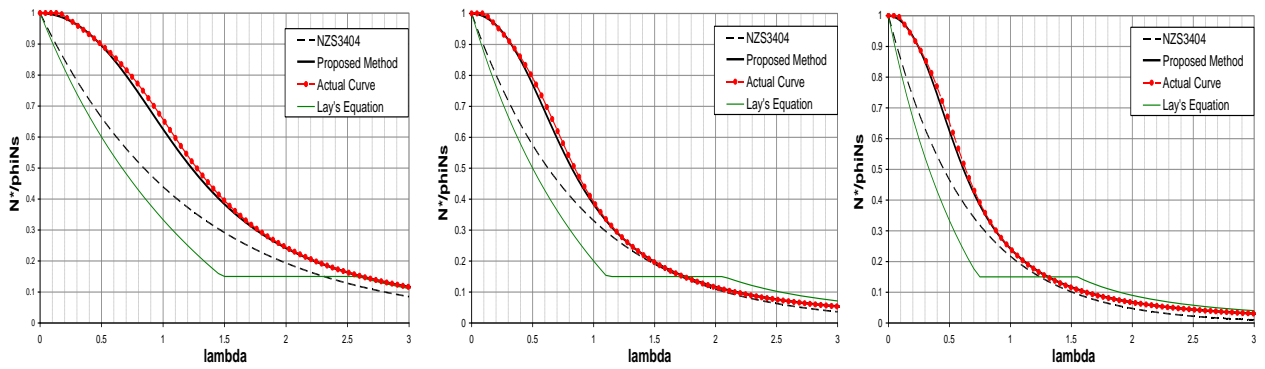
(c)  $\beta = 0$



(d)  $\beta = -0.5$

(e)  $\beta = -1$

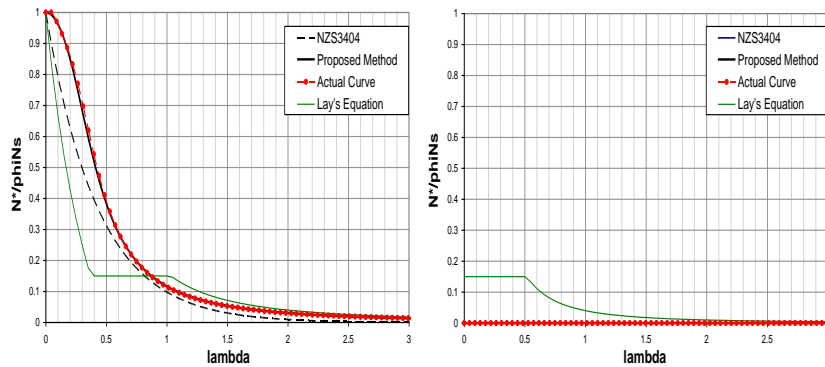
**Figure 9: Comparison of proposed EYC procedure for  $\alpha_b = -1$ .**



(a)  $\beta = 1$

(b)  $\beta = 0.5$

(c)  $\beta = 0$



(d)  $\beta = -0.5$

(e)  $\beta = -1$

**Figure 10: Comparison of proposed EYC procedure for  $\alpha_b = 0$ .**

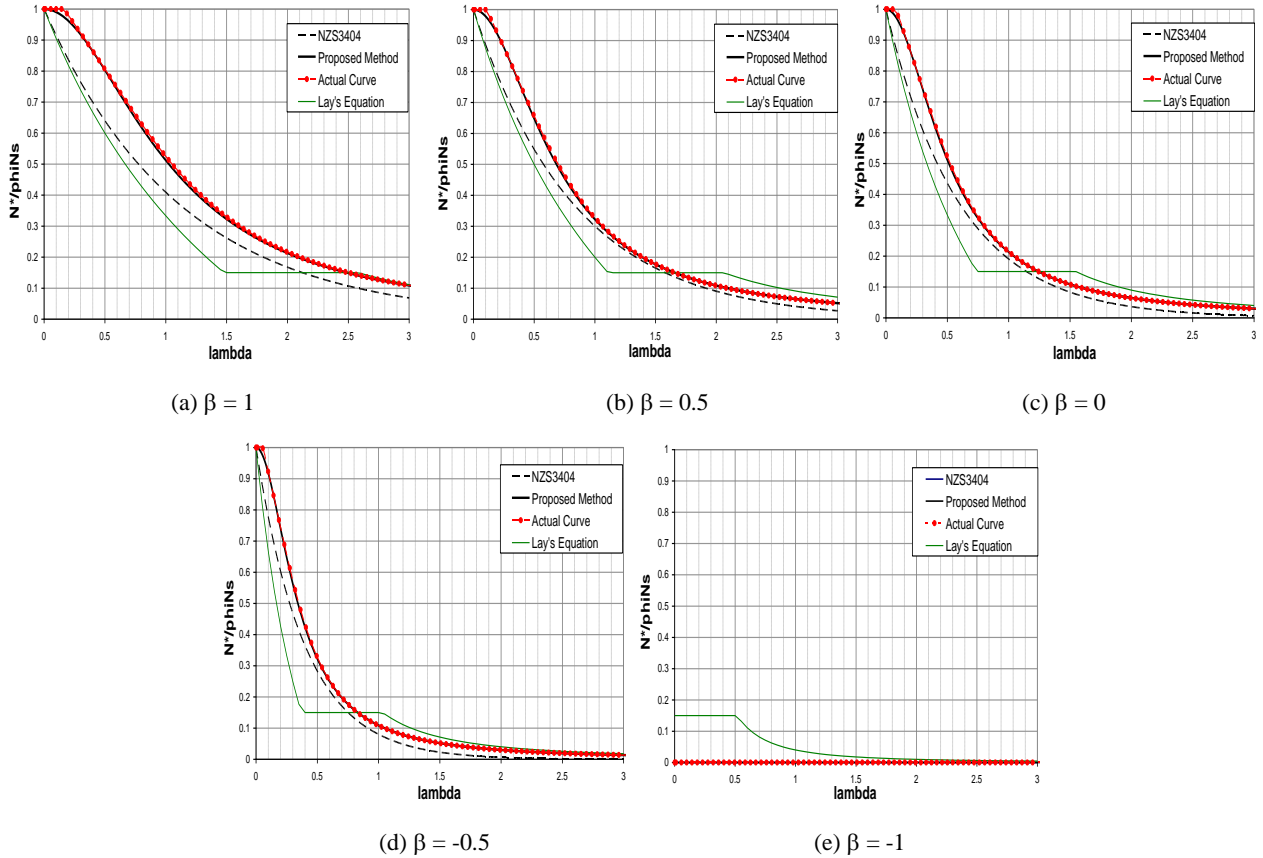


Figure 11: Comparison of proposed EYC procedure for  $\alpha_b = 1$ .

## 5. COMPARISON WITH EXISTING END YIELD CRITERIA

Graphs are shown for the cases of  $\alpha_b = -1.0, 0.0$  and  $1.0$  in Figures 9, 10 and 11. Graphs for other  $\alpha_b$  values may be found in Lu [3]. It may be seen that the proposed curves, which use the Equation 10 approximation, follow the actual curves (which use the exact SRF) very well. It is much less conservative than the current NZS3404 (SNZ, 2007) and Lay's Equation used in AS 4100(SA, 1998) for small values of  $\lambda$  which represent many actual columns.

## 6. CONCLUSIONS

Revised equations were developed and proposed for the end yielding criteria (EYC) of columns in steel frames. The equations are developed considering residual stress effects and stability effects independently, and then combined. This results in simpler and less conservative equations than those in current steel structures standards.

## ACKNOWLEDGEMENTS

The authors would like to acknowledge the financial support provided by the Steel Construction New Zealand, SCNZ, Heavy Engineering Research Association, HERA, and the University of Canterbury.

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