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Lateral dynamic soil stiffness for partially embedded foundations in heterogeneous soils

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ABSTRACT: The first author has recently developed the differential equation cell method to formulate the dynamic soil stiffnesses for partially embedded foundations in homogeneous soils. In this paper, this method is further extended to foundations in heterogeneous soils. The expressions for soil responses are obtained in simple closed forms, and the computation process requires iterations. The developed formulation for the lateral stiffness is demonstrated for computations of rigid foundations partially embedded in heterogeneous soils. The computations are found to generally converge with very little iteration for the cases analyzed. The developed final expression for the stiffness is simple yet produces the results very close to those computed by the much more elaborated method.

1 INTRODUCTION

Simple yet rational formulations were developed for dynamic response analysis of foundations. Those developed by Novak and his colleagues (e.g. Beredugo & Novak 1972) treated the soil at the side of foundation as a stack of mutually uncoupled thin homogeneous layers. The stiffnesses of an individual layer at the foundation were formulated from vibrations of a horizontal, massless, rigid, circular slice of the foundation contained in a horizontal layer of unit thickness. Later, those for time-domain analysis were also developed (Nogami et al. 1988, 1991a). A nonlinear mechanism was introduced further in the side soil stiffnesses (Nogami et al. 1991b, 1992a).

The differential equation cell method was proposed to develop simple formulations for dynamic responses of partially embedded foundations. In this method, the soil medium is treated as an assembly of a number of cells, in which each cell contains two differential equations. These equations are coupled between those in adjacent cells and are solved in a convenient manner. This method was used for foundations in homogeneous soils (Nogami et al. 2001, 2002). The present paper extends the approach further to foundations in heterogeneous soils.

2 FORMULATION

2.1 Differential equations for fundamental cell of heterogeneous medium

A rectangular cell, as shown in Fig. 1, is considered in the medium. The elastic parameters are assumed to increase linearly with depth. Then, the complex Lame constants are expressed as

$$G^*(\zeta) = (1+2Di)G(\zeta) = (1+2Di)\big(G(\zeta_a) + C_G(\zeta-\zeta_a)\big) \qquad (1a)$$

$$\lambda^*(\zeta) = (1+2Di)\lambda(\zeta) = (1+2Di)\big(\lambda(\zeta_a) + C_\lambda(\zeta-\zeta_a)\big) \qquad (1b)$$

where $G(\zeta)$ and $\lambda(\zeta)$ = Lames's constants at ζ; ζ_a and ζ_b = upper and lower ends of the cell in the ζ coordinate, respectively; D = material damping parameter; and C_G and C_λ = constants mutually related

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through Poison’s ratio given by

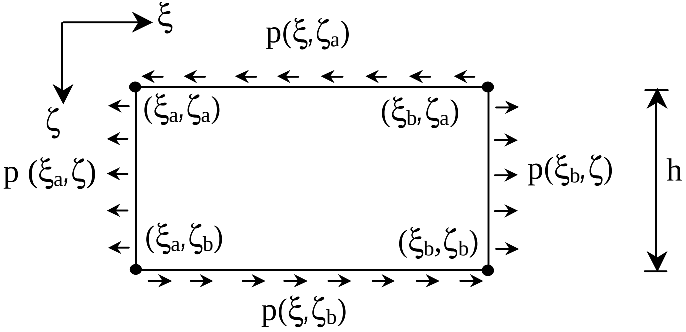

Fig.1 Fundamental cell subjected to tractions

$$\lambda(\zeta)=\frac{\nu}{(1-2\nu)}G(\zeta) \tag{2}$$

Neglecting the vertical displacement, the equation of horizontal motion of the medium is written in the frequency domain. Its slightly modified form is

$$\frac{\partial}{\partial\xi}\left(k_s(\zeta)\frac{\partial u(\xi,\zeta)}{\partial\xi}\right)+\frac{\partial}{\partial\zeta}\left(k_c(\zeta)\frac{\partial u(\xi,\zeta)}{\partial\zeta}\right)+m_c\omega^2 u(\xi,\zeta)=0 \tag{3}$$

where u = displacement amplitude; ω = circular frequency; and denoting ρ = unit mass of soil,

$$k_c(\zeta)=G^*(\zeta)\bar{k}_c \tag{4a}$$

$$k_s(\zeta)=\left(\lambda^*(\zeta)+2G^*(\zeta)\right)\bar{k}_s \tag{4b}$$

$$m_c=\rho\bar{m}_c \tag{4c}$$

with $\bar{k}_c$, $\bar{k}_s$ and $\bar{m}_c$ = non-dimensional parameters. It is noted that Eq. 3 is also the equation of motion of a column in a system of closely spaced columns that are interconnected by distributed lateral springs along the side (Nogami et al., 1990). k_c and k_s correspond respectively to the complex column stiffness in shear and the complex spring stiffness, and m_c corresponds to the mass per unit length of the column. These are related uniquely only with the material properties of the original continuous medium through Eqs. 4a~4c, and $\bar{k}_c$ and $\bar{k}_s$ are dependent only on ν (Nogami et al., 1990).

The displacement is assumed to be expressed as

$$u(\xi,\zeta)=X(\xi)Z(\zeta) \tag{5}$$

where X(0) = 1. Substituting Eq. 5 into Eq. 3 and denoting $\phi(\xi)$ as a weight function, the Galerkin method for weighted residual over ξ in the cell yields

$$\left(k_s(\zeta)\int_{\xi_a}^{\xi_b}\frac{d^2X(\xi)}{d\xi^2}\phi(\xi)d\xi\right)Z(\zeta)+\frac{d}{d\zeta}\left\{\left(k_c(\zeta)\int_{\xi_a}^{\xi_b}X(\xi)\phi(\xi)d\xi\right)\frac{dZ(\zeta)}{d\varsigma}\right\}+$$

$$\left(m_c\int_{\xi_a}^{\xi_b}X(\xi)\phi(\xi)d\xi\right)Z(\zeta)=0 \tag{6}$$

Integrating the first term by parts, Eq. 6 results in

$$-\frac{d}{d\zeta}\left(n(\zeta)\frac{dZ(\zeta)}{d\zeta}\right)+\left(k(\zeta)-m\omega^2\right)Z(\zeta)=\phi(\xi_b)p(\xi_b,\zeta)-\phi(\xi_a)p(\xi_a,\zeta) \tag{7}$$

where ξ_a and ξ_b = left and right ends of the cell in the ξ coordinate, respectively; $p(\xi_{a,b}, \zeta)$ is the traction acting at $(\xi_{a,b}, \zeta)$ expressed as

$$p(\xi_{a,b}, \zeta) = -k_s(\zeta)\left.\frac{dX(\xi)}{d\xi}\right|_{\xi_{a,b}} Z(\zeta) \tag{8}$$

and

$$n(\zeta) = k_c(\zeta)\int_{\xi_a}^{\xi_b} X(\xi)\phi(\xi)d\xi \tag{9a}$$

$$k(\zeta) = k_s(\zeta)\int_{\xi_a}^{\xi_b} \frac{dX(\xi)}{d\xi}\frac{d\phi(\xi)}{d\xi}d\xi \tag{9b}$$

$$m = m_c\int_{\xi_a}^{\xi_b} X(\xi)\phi(\xi)d\xi \tag{9c}$$

Similarly, substituting Eq. 5 into Eq. 3 and using $?(\zeta)$ as a weight function, the Galerkin method for weighted residual over ζ in the cell yields

$$-N(\zeta)\frac{d^2X(\xi)}{d\xi^2} + \left(K(\zeta) - M\omega^2\right)X(\xi) = \psi(\zeta_b)p(\xi, \zeta_b) - \psi(\zeta_a)p(\xi, \zeta_a) \tag{10}$$

where $p(\xi, \zeta_{a,b})$ = traction acting at $(\xi, \zeta_{a,b})$; and

$$N = \int_{\zeta_a}^{\zeta_b} k_s(\zeta)Z(\zeta)\psi(\zeta)d\zeta \tag{11a}$$

$$K = \int_{\zeta_a}^{\zeta_b} k_c(\zeta)\frac{dZ(\zeta)}{d\psi}\frac{d\psi(\zeta)}{d\zeta}d\zeta \tag{11b}$$

$$M = m_c\int_{\zeta_a}^{\zeta_b} Z(\zeta)\psi(\zeta)d\zeta \tag{11c}$$

Eqs. 7 and 10 are the fundamental differential equations for a cell. The weight functions are selected as $\phi(\xi) = X(\xi)$ and $?(\zeta) = Z^*(\zeta)$, in which $Z^*(z)$ is the conjugator of $Z(z)$ (Nogami et al. 2002).

2.2 **Differential equations for secondary cell of layered medium**

A secondary cell is assumed to contain j heterogeneous fundamental cells as shown in Fig. 2. The coordinate ξ is assumed to be located at the left end of the secondary cell (i.e. $x = \xi$). The compatibility condition between the j^{th} and $j+1^{th}$ fundamental cells requires

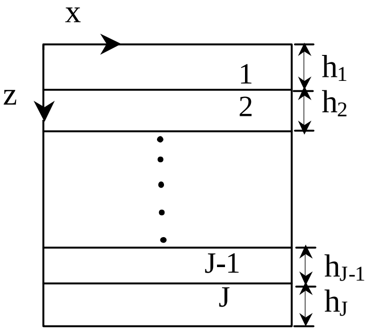

Fig.2 Secondary cell made of fundamental cells

$$X(x)_j = X(x)_{j+1} \tag{12a}$$

$$Z(\zeta_b)_j = Z(\zeta_a)_{j+1} \tag{12b}$$

and the equilibrium condition between the jth and j+1th fundamental cells does

$$p(x,\zeta_b)_j - p(x,\zeta_a)_{j+1} = 0$$

or

$$\sum_{j=1}^{J}\left\{\psi(\zeta_b)_j\, p(x,\zeta_b)_j - \psi(\zeta_a)_j\, p(x,\zeta_a)_j\right\} = \psi(\zeta_b)_J\, p(x,\zeta_b)_J - \psi(\zeta_a)_1\, p(x,\zeta_a)_1 \tag{13}$$

where j = layer number. Thus, with Eqs. 12a, 12b and 13, Eqs. 7 and 10 for the fundamental cell lead to the differential equations for the secondary cell as, respectively

$$-\frac{d}{dz}\left(n(\zeta)_j\,\frac{dZ(\zeta)_j}{d\zeta}\right) + \left(k(\zeta)_j - m\omega^2\right)Z(\zeta)_j = \phi(x_b)p(x_b,\zeta)_j - \phi(x_a)p(x_a,\zeta)_j \qquad j = 1\sim J \tag{14a}$$

$$-\sum_{j=1}^{J} N_j\,\frac{d^2X(x)}{dx^2} + \sum_{j=1}^{J}\left(K_j - M_j\omega^2\right)X(x) = \psi(\zeta_b)_J\, p(x,\zeta_b)_J - \psi(\zeta_a)_1\, p(x,\zeta_a)_1 \tag{14b}$$

2.3 **Boundary value problem**

A partially embedded rigid foundation is considered in the inhomogeneous soil as shown in Fig. 3. Only the shaded area in the figure is considered for formulation. Soil medium around the foundation is divided into three secondary cells (Cells I, II and III) as shown in Fig. 3. The foundation is assumed to undergo the lateral translational motion of amplitude U.

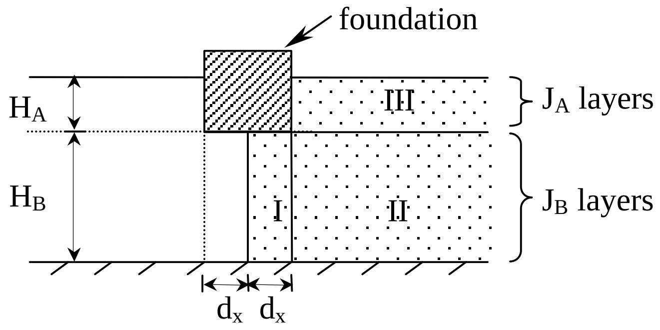

Fig.3 Soil medium divided into three secondary cells

2.3.1 *Functions X(x) and Z(z)*

The boundary conditions for each secondary cell are

$$\text{Cell I}\ \begin{cases} X(x) = 1 \\ Z(\zeta_a)_1 = U \quad \text{and} \quad Z(\zeta_b)_{J_B} = 0 \end{cases} \tag{15a}$$

$$\text{Cell II}\ \begin{cases} X(0) = 1 \ \text{and}\ X(\infty) = 0 \\ Z(\zeta_a)_1 = U \quad \text{and} \quad Z(\zeta_b)_{J_B} = 0 \end{cases} \tag{15b}$$

$$\text{Cell III}\ \begin{cases} X(0) = 1 \quad \text{and} \quad X(\infty) = 0 \\ Z(\zeta)_j = U \qquad j = 1 \sim J_A \end{cases} \tag{15c}$$

where J_A = number of fundamental cells in Cell III; and J_B = number of fundamental cells in Cells I and II. In addition, $p(0, \zeta)_j$ for $j = 1 \sim J_B$ in Cell I and $p(x, \zeta_a)$ in Cell III are zero.

Adding Eq. 14a for Cell I and Eq. 14a for Cell II together and applying the boundary conditions at the interface between Cells I and II lead to

$$-\frac{d}{d\zeta}\left(n'(\zeta)_j \frac{dZ(\zeta)_j}{d\zeta}\right) + \left(k'(\zeta)_j - m'\omega^2\right)Z(\zeta)_j = 0 \qquad j = 1\sim J_B \tag{16a}$$

and Eq. 14b for Cells II and III lead similarly to

$$-\sum_{j=1}^{J_C} N_j \frac{d^2X(x)}{dx^2} + \sum_{j=1}^{J_C}\left(K_j - M_j\omega^2\right)X(x) = 0 \tag{16b}$$

where $n' = n^I + n^{II}$; $k' = k^I + k^{II}$; $m' = m^I + m^{II}$; and $J_C = J_A + J_B$. With the boundary conditions associated with X(x) for Cells II and III, the solution of Eq. 16b is expressed as

$$X(x)^{II,III} = e^{-\beta x} \tag{17a}$$

and, using the polynomial form, the general solution of Eq. 16a is

$$Z(\zeta)_j^{I,II} = \left(1 + \sum_{n=2,3,4,\cdots} \alpha_{n\,j}^a \zeta^n\right)a_j + \left(\zeta + \sum_{n=2,3,4,\cdots} \alpha_{n\,j}^b \zeta^n\right)b_j \tag{17b}$$

where a_j and b_j = unknown constants to be defined later; and

$$\beta = \sqrt{\left(\sum_{j=1}^{J_C} K_j - \omega^2 M_j\right) \Big/ \left(\sum_{j=1}^{J_C} N_j\right)} \tag{18a}$$

$$\alpha_{n\,j}^a = \frac{C_j\alpha_{n-3,\,j}^a + D_j\alpha_{n-2,\,j}^a - (n-1)^2 A_j \alpha_{n-1,\,j}^a}{n(n-1)B_j} \quad (C_j = 0 \text{ for } n = 2\sim 4 \text{ and } A_j = 0 \text{ for } n = 2) \tag{18b}$$

$$\alpha_{n\,j}^b = \frac{C_j\alpha_{n-3\,j}^b + D_j\alpha_{n-2\,j}^b - (n-1)^2 A_j \alpha_{n-1,\,j}^b}{n(n-1)B_j} \quad (C_j = 0 \text{ for } n = 2 \text{ \& } 3 \text{ and } D_j = 0 \text{ for } n = 2) \tag{18c}$$

$$A_j = \left\{n'(\zeta_b)_j - n'(\zeta_a)_j\right\}/h_j \tag{18d}$$

$$B_j = n'(\zeta_a)_j \tag{18e}$$

$$C_j = \left\{k'(\zeta_b)_j - k'(\zeta_a)_j\right\}/h_j \tag{18f}$$

$$D_j = k'(\zeta_a)_j - \omega^2 m'_j \tag{18g}$$

where h_j = thickness of the j^{th} layer. Imposing compatibility between the soil and foundation to Eqs. 14a for Cell III and to X(x) for Cell I, the rest of the functions for the secondary cells are defined as: respectively

$$\left(k(\zeta)_j - m\omega^2\right)Z(\zeta)_j^{III} = -p(0,\zeta)_j^{III} \qquad j = 1 \sim J_A \tag{19a}$$

$$X(x)^I = 1 \tag{19b}$$

2.3.2 *Constants a_j and b_j*

The conditions for $Z(\zeta)_j$ in Cells I and II can be stated as

$$Z(\zeta_a)_1 = U \tag{20a}$$

$$\begin{Bmatrix} Z(\zeta_b) \\ \dot{Z}(\zeta_b) \end{Bmatrix}_j = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{n(\zeta_a)_{j+1}}{n(\zeta_b)_j} \end{bmatrix} \begin{Bmatrix} Z(\zeta_a) \\ \dot{Z}(\zeta_a) \end{Bmatrix}_{j+1} \tag{20b}$$

$$Z(\zeta_b)_{J_B} = 0 \tag{20c}$$

With $(\zeta_a , \zeta_b)_j = (0 , h_j)$ and $(a , b)_j = a_{JB}(a' , b')_j$, substituting Eq. 17b into Eqs. 20b and 20c result in, respectively

$$\begin{Bmatrix} a' \\ b' \end{Bmatrix}_j = \frac{1}{\Delta_j} \begin{bmatrix} 1+\sum\limits_{n=2} n\alpha_n^b h^{n-1} & -\left(h+\sum\limits_{n=2} \alpha_n^b h^n \right) \\ -\left(\sum\limits_{n=2} n\alpha_n^a h^{n-1} \right) & 1+\sum\limits_{n=2} \alpha_n^a h^n \end{bmatrix}_j \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{n(0)_{j+1}}{n(h)_j} \end{bmatrix} \begin{Bmatrix} a' \\ b' \end{Bmatrix}_{j+1} \tag{21a}$$

$$\begin{Bmatrix} a' \\ b' \end{Bmatrix}_{J_B} = \begin{Bmatrix} 1 \\ -\dfrac{1+\sum\limits_{n=2} \alpha_n^a h^n}{h+\sum\limits_{n=2} \alpha_n^b h^n} \end{Bmatrix}_{J_B} \tag{21b}$$

where

$$\Delta_j = \left(1+\sum_{n=2} \alpha_n^a h^n \right)_j \left(1+\sum_{n=2} n\alpha_n^b h^n \right)_j - \left(h+\sum_{n=2} \alpha_n^b h^n \right)_j \left(\sum_{n=2} n\alpha_n^a h^{n-1} \right)_j \tag{22}$$

Therefore, starting with $(a' , b')_{JB}$ given by Eq. 21b, $(a' , b')_j$ can be computed from $j = J_B–1$ through 1 successively by Eq. 21a. After $(a', b')_1$ is computed, a_{JB} is obtained to satisfy Eq. 20a at the top ($j = 1$) in Cells I and II. Then, $(a , b)_j$ is computed from $(a , b)_j = a_{JB}(a' , b')_j$.

2.4 **Dynamic stiffness for partially embedded foundation**

When the force P is applied to the foundation, the equilibrium condition at the foundation is stated as

$$\begin{aligned} P &= -2\left\{ \left. \left(n(\zeta)_1 \frac{\partial u(d_x , \zeta)_1}{\partial \zeta} \right)^{I} \right|_{\zeta=0} + \left. \left(n(\zeta)_1 \frac{\partial u(0, \zeta)_1}{\partial \zeta} \right)^{II} \right|_{\zeta=0} + \left(\sum_{j=1}^{J_A} \int_0^{h_j} p(0, \zeta)_j \, d\zeta \right)^{III} \right\} \\ &= -2\left\{ \left. \left(n(\zeta)_1 \frac{dZ(\zeta)_1}{d\zeta} \right)^{I} \right|_{\zeta=0} + \left. \left(n(\zeta)_1 \frac{dZ(\zeta)_1}{d\zeta} \right)^{II} \right|_{\zeta=0} + \left(\sum_{j=1}^{J_A} \int_0^{h_j} \left(k(\zeta)_j - \omega^2 m \right) Z(\zeta)_j \, d\zeta \right)^{III} \right\} \end{aligned} \tag{23}$$

Evaluating the above expression with Eqs. 9a~9c and U = 1, the dynamic soil stiffness for this foundation (K_f) is

$$K_f = -2\left[n'(0)_1 \right]^{I,II} - 2\left[\sum_{j=1}^{J_B} \frac{1}{2} \left\{ k(0)_j + k(h)_j \right\} h_j - \omega^2 m h_j \right]^{III}$$

$$= -2\left[k_c(0)_1\left(d_x + \frac{1}{\beta}\right)\right]^{I,II} - \left[\sum_{j=1}^{J_A} \frac{\beta}{2}\left\{k_s(0)_j + k_s(h)_j\right\}h_j - \omega^2 \frac{m_c}{\beta} h_j\right]^{III} \qquad (24)$$

where d_x = a half width of the foundation.

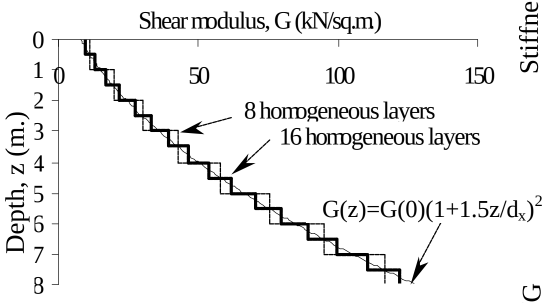

Fig.4 Distribution of shear modulus with depth replaced by 8 and 16 homogeneous layers

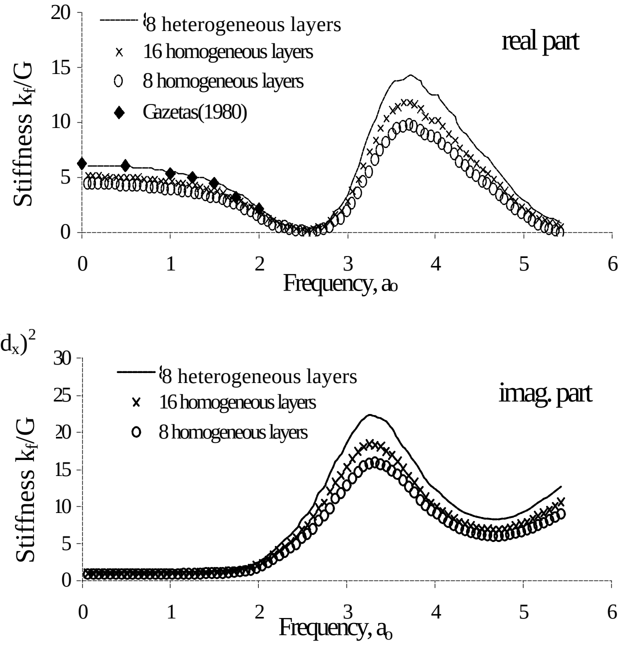

Fig.5 Dynamic soil stiffnesses computed for three different cases

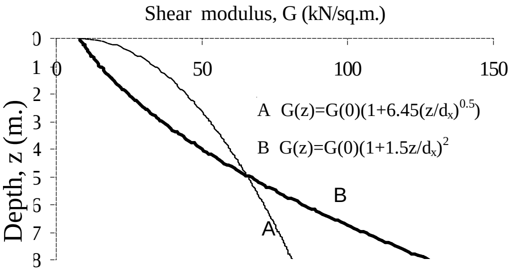

Fig.6 Distribution of shear modulus with depth for profile A and Profile B

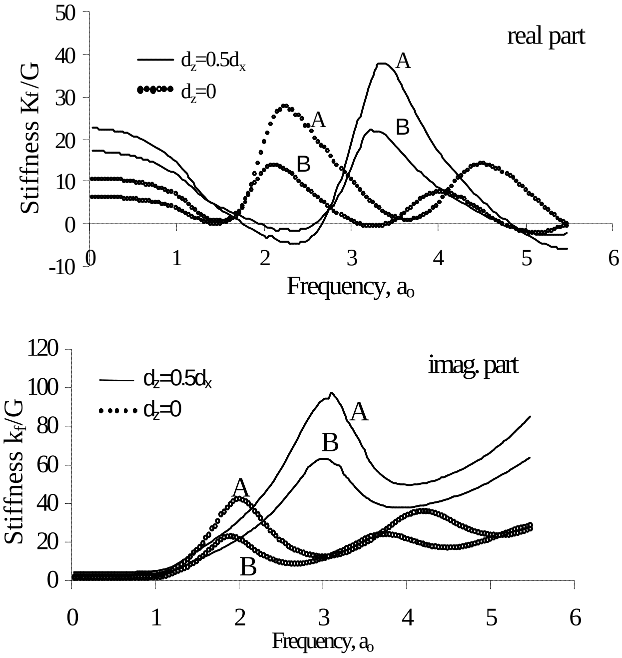

Fig.7 Dynamic soil stiffnesses for embedded and surface foundations in profile A and profile B soils

3 COMPUTED RESULTS

The dynamic soil stiffnesses for rigid foundations are computed by the expression given by Eq. 24. All parameters in the expression are provided as inputs except β. The parameter β to define X(x) and the constants $(a, b)_j$ to define $Z(\zeta)_j$ are mutually coupled in the formulation. Thus they are calculated iteratively. In the computations carried out below, the convergence in the iterations was achieved generally within 8 iterations for tolerance 1%.

A foundation of d_x = 4m is assumed to rest on the surface of soil underlain by a rigid base at depth $2d_x$. The conditions considered for the soil are: $v_s(z) = v_s(0)(1+1.5z/d_x)$ with v_s = shear wave velocity of soil (or $G(z) = G(0)(1+1.5z/d_x)^2$); $\nu = 0.25$; and D = 0.05. The soil is divided equally into 8 or 16 homogeneous layers as shown in Fig. 4. It is also divided equally into eight heterogeneous layers in which G(z) varies linearly with z within a layer. G(z) in the latter distribution is nearly equal to the original G(z). Soil stiffnesses computed for these three cases are shown in Fig. 5 together with those computed by the far more elaborated method (Gazetas 1980): $a_0 = \omega r_0/v_s(0)$ in Fig. 5. It is seen in the figure that the capability of handling heterogeneity of a subdivided layer increases accuracy significantly.

The foundation without or with embedment ($d_z = 0$ or $0.5d_x$) is considered in two heterogeneous soil profiles (Fig. 6), in which Profile A is defined as $G(z) = G(0)(1+6.45(z/d_x)^{0.5})$ and Profile B is the one considered above. Both G(0) and the average G over the depth are identical between the two profiles. The computed soil stiffnesses for a foundation with or without embedment in this soil are shown in Fig. 7. It is seen that the embedment affects the dynamic soil stiffness, not only its magnitude, but also its frequency dependency.

4 CONCLUSIONS

The differential equation cell method enables us to formulate the soil stiffness, for partially embedded foundations in heterogeneous soils, in a simple closed form. It requires iterations in computation. Sufficient convergence is generally observed within very few iterations for the cases computed herein. The developed formulation can calculate the dynamic stiffness very close to that computed by far more elaborated methods.

REFERENCES:

Beredugo Y.O. and Novak M. 1972. A coupled horizontal and rocking vibration of embedded footings. *Canadian Geotechnical Journa.* 9. 477-497.

Gazetas G. 1980. Static and dynamic displacements of foundations on heterogeneous multilayered soils,” *Geotechnique*, 30 (2),159-177.

Nogami T. and Konagai K. 1988. Time-domain flexural response of dynamically loaded single piles. *J. Engrg. Mech.* 114 (9). 1512-1525: ASCE.

Nogami, T. and Leung, M.B. 1990. A simplified mechanical subgrade model for dynamic response analysis of shallow foundations. *Int. J. Earthq. Engrg. Struct. Dyn.* 19. 1041-1055.

Nogami T., Konagai K. and Otani J. 1991a. Time domain axial response of dynamically loaded single pile. *J. Engrg. Mech.* 112 (2). 1241-1252: ASCE.

Nogami T., Konagai K. & Otani J. 1991b. Nonlinear time domain numerical model for pile group under transient dynamic forces. Proc. *Second Intern. Conf. On Recent Adv. In Geo. Engrg. and Soil Dyn.* St. Louis, MO Paper No. 5.51. 881-888.

Nogami T., Otani J., Konagai K. and Chen H.C. 1992a. Nonlinear soil-pile interaction for dynamic lateral motion. *J. Geotech. Engrg.* 118 (1). 89-105: ASCE.

Nogami T., Mikami A. and Konagai K. 2001. Simple formulation of ground impedance functions for rigid surface foundations, *Soil Dynamics and Earthquake Engineering*. 21. 475-484.

Nogami T., Chen H.S. 2002. Boundary differential equation method: simplified dynamic soil stiffnesses for embedded rigid foundations. *Soil Dynamics and Earthquake Engineering.* 22. 323-334.