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# A new algorithm for non-linear dynamic structural analysis

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**ABSTRACT:** The analysis of structures under dynamic loading is generally carried out using time stepping methods, of which the most popular is Newmark's method. In most non-linear methods, dynamic equilibrium is satisfied at the ends of the time step, although in some methods it is satisfied at the midpoint. In this paper, linear structures are first considered, and properties of the exact amplification matrix are noted. An approximate procedure is then developed using cubic interpolation, with coefficients being chosen using weighted residuals with unspecified weight functions. Properties of the weight functions are then determined such that the amplification matrix is of a similar form to the exact matrix, and are unconditionally stable, permits controlled algorithmic damping and is of the maximum available accuracy. The procedure is shown to be equivalent to a weighted sum procedure based on two pairs of sampling points, thus satisfying dynamic equilibrium in an average sense. The weighted sum procedure is ideal for solving non-linear problems.

## 1 INTRODUCTION

In structural dynamics, the equations of motion for a spatially discretised multi-degree of freedom (MDOF) linear system are of the form

$$[M]\{a(t)\}+[C]\{v(t)\}+[K]\{x(t)\}-\{F(t)\}=\{0\} \tag{1}$$

where $[M]$, $[C]$ and $[K]$ are the mass, damping and stiffness matrices, $\{a(t)\}$, $\{v(t)\}$ and $\{x(t)\}$ are acceleration, velocity and displacement vectors, $\{F(t)\}$ is a prescribed force vector and $t$ is time. If there are $N$ degrees of freedom, $[M]$, $[C]$ and $[K]$ are ($N$ x $N$) symmetrical matrices, and $\{a(t)\}$,$\{v(t)\}$, $\{x(t)\}$ and $\{F(t)\}$ are ($N$ x $1$) vectors. If there are material or geometric nonlinearities, the governing equation is

$$[M]\{a(t)\}+[C]\{v(t)\}+\{F_S(x(t))\}-\{F(t)\}=\{0\} \tag{2}$$

where $\{F_S(x(t))\}$ are internal forces which are functions of displacement. Equation 1 or 2 is to be solved subject to the initial conditions that at time $t=0$, the displacement and velocity vectors $\{x(0)\}$ and $\{v(0)\}$ are prescribed.

Exact solutions may be obtained for the linear case, which involve the determination of *2N* generally complex eigenvalues and corresponding eigenvectors (see for example Meirovitch (1980)). With non-linear cases, numerical time stepping procedures are adopted. Many time stepping algorithms are available with a number of these being discussed by Hughes and Belytshko (1983) and Dokainish and Subbaraj (1989a and 1989b), and Fung has recently developed a range of time stepping procedures based on Padé approximations of the exact solution of single degree of freedom (SDOF) systems (Fung 1999a, 1999b). An extensive mathematical treatment is given in the text by Wood (1990). In single-step two-stage algorithms, the displacement and velocity are specified at time $t=0$ and are estimated during the period $0<t<=\Delta t$ where $\Delta t$ is the time step. Values of displacement and velocity are calculated at $t=\Delta t$, and become the initial values for the next time step. The most popular time stepping method is Newmark's method (Newmark 1959). At the recent Twelfth World Congress on Earthquake Engineering, Newmark's method was the procedure used for the numerical solution of Equation 1 in fifty seven papers. Wood (1990) has carried out extensive studies on Newmark's method and the θ method (not to be confused with the Wilson θ method (Bathe and Wilson (1973)), and as a mathematician lists four reasons for the popularity of Newmark's method, the last being that "It has been in use since 1959".

In this paper, the exact amplification matrix used with single-step two-stage algorithms is presented in a form thought to be new, which is shown to have a number of properties which may be used to reduce computation time. The single degree of freedom solution is then derived in a similar form, and some interesting properties of the amplification matrix are observed. An unconditionally stable approximate method is then developed using cubic polynomials as interpolation functions. A weighted residual procedure using two unspecified weight functions is followed. "Areas" and "moments of area" of the weight functions are selected such that the algorithm has the following features:

- The approximate amplification matrix is of a form similar to the exact amplification matrix.
- Algorithmic damping may be incorporated by specifying a single parameter.
- The algorithm has maximum accuracy.
- Numerical operations are minimised.

Although the weight functions are not specified, equivalent weight functions consisting of two Dirac delta functions are also determined. Use of these latter functions is equivalent to a weighted sum method, which is then applied to solve non-linear problems.

## 2 EXACT AMPLIFICATION MATRIX

The displacement and velocity at time $t$ may be expressed in terms of values at time $t=0$ by the equation

$$\begin{bmatrix} \{x(t)\} \\ \{v(t)\}t \end{bmatrix} = \begin{bmatrix} [A_{11}(t)] & [A_{12}(t)] \\ [A_{21}(t)] & [A_{22}(t)] \end{bmatrix} \begin{bmatrix} \{x(0)\} \\ \{v(0)\}t \end{bmatrix} + \begin{bmatrix} \{F\}_1 \\ \{F_2\} \end{bmatrix} \tag{3}$$

The matrix

$$[A(t)] = \begin{bmatrix} [A_{11}(t)] & [A_{12}(t)] \\ [A_{21}(t)] & [A_{22}(t)] \end{bmatrix} \tag{4}$$

is the *amplification matrix.* With linear problems, $[A(t)]$ may be determined exactly (Meirovitch (1980)). The solution involves the introduction of $N$ subsidiary equations, and requires the determination of $2N$ eigenvalues and eigenvectors. The eigenvectors have two important orthogonality properties, which are used to uncouple the equations. Initial conditions are generally incorporated by using one of the othogonality conditions. If both are considered however, it is possible to show that the amplification matrix may be expressed in the form

$$[A^e(t)] = \begin{bmatrix} [A^e_{11}(t)] & [A^e_{12}(t)] \\ [A^e_{21}(t)] & [A^e_{22}(t)] \end{bmatrix} = \begin{bmatrix} [R^e_2(t)][C] + [R^e_3(t)][M] & \frac{1}{t}[R^e_2(t)][M] \\ -t[R^e_2(t)][K] & [R^e_3(t)][M] \end{bmatrix} \quad (5)$$

where the superscript *e* indicates the solution is exact. The matrices $[R_2(t)]$ and $[R_3(t)]$ are symmetrical. In the SDOF case, the amplification matrix is

$$[A^e(t)] = \begin{bmatrix} R^e_2(t)c + R^e_3(t)m & \frac{1}{t}R^e_2(t)m \\ -tR^e_2(t)k & R^e_3(t)m \end{bmatrix} \quad (6)$$

where *m*, *c* and *k* are mass, damping coefficient and stiffness of the SDOF system. In the SDOF case the matrices $[R_2(t)]$ and $[R_3(t)]$ become scalars $R_2(t)$ and $R_3(t)$ which may be determined analytically as

$$R^e_2(t) = \frac{2}{\sqrt{4km - c^2}} \sin\left(\frac{\sqrt{4km - c^2}}{2m} t\right) e^{-ct/2m} \quad (7)$$

and

$$R^e_3(t) = \frac{1}{m}\left(\cos\left(\frac{\sqrt{4km - c^2}}{2m} t\right) - \frac{c}{\sqrt{4km - c^2}} \sin\left(\frac{\sqrt{4km - c^2}}{2m} t\right)\right) e^{-ct/2m} \quad (8)$$

Important properties of any approximate method are obtained from the eigenvalues of the amplification matrix. It should be noted that the eigenvalues of the amplification matrix given by Equation 6 are

$$\frac{1}{2}(R_2(t)c + 2R_3(t)m) \pm \frac{1}{2}R_2(t)\sqrt{4km - c^2} \quad (9)$$

for any functions $R_2(t)$ and $R_3(t)$. Thus any approximate amplification matrix which can be written in the form of Equation 6 has complex eigenvalues if $c^2\text{-}4km<0$, ie the system is underdamped, and has real eigenvalues if $c^2\text{-}4km>=0$. The resulting solution using such an amplification matrix therefore automatically produces an oscillating displacement for underdamped systems, and a non-oscillating displacement for overdamped systems. The single-step two-stage form of Newmark's method cannot be written in the form of Equation 6 with general parameters $\beta$ and $\gamma$.

## 3 APPROXIMATE AMPLIFICATION MATRIX

In this section, an approximate algorithm is briefly described. The displacement of a SDOF system, $x(t)$, is approximated by a cubic function. Two coefficients are chosen to satisfy the initial conditions, and the remaining two coefficients are chosen to satisfy two weighted integral equations with unspecified weight functions (for a description of this method with first order problems see Zienkiewicz (1977)). The approximate amplification matrix is then determined, with lengthy algebraic terms of the form

$$A^a_{ik} = A^a_{ik}(m, c, k, \Delta t, u_{10}, u_{11}, u_{12}, u_{13}, u_{20}, u_{21}, u_{22}, u_{23}) \quad (10)$$

where $\Delta t$ is the time step, and the coefficients $u_{ij}$ are "areas" and "moments of area" of the two weight functions. The eight coefficients are now selected in the following series of steps.

### 3.1 Establish desirable form of the amplification matrix

Solve the four equations implied by

$$[A^a(\Delta t)] = \begin{bmatrix} R_2^a(\Delta t)c + R_3^a(\Delta t)m & \frac{1}{\Delta t}R_2^a(\Delta t)m \\ -\Delta t R_2^a(\Delta t)k & R_3^a(\Delta t)m \end{bmatrix} \quad (11)$$

for two coefficients and the two approximate functions $R_2{}^a(\Delta t)$ and $R_3{}^a(\Delta t)$.

### 3.2 **Increase order of accuracy of the algorithm**

Eliminate terms $O(\Delta t^3)$ in the expansions of the differences $R_2{}^a(\Delta t)$ - $R_2{}^e(\Delta t)$ and $R_3{}^a(\Delta t)$ - $R_3{}^e(\Delta t)$.

### 3.3 **Incorporate algorithmic damping**

This is achieved by introducing a damping parameter $\rho_b$ which is the spectral radius as $\Delta t$ tends to infinity. Wood (1990) suggests this should be in the range 0.5-0.7 to algorithmically damp high frequencies. If $\rho_b$=1 there is no algorithmic damping, and if $\rho_b$=0 there is algorithmic annihilation. It is also convenient to introduce a related parameter

$$b = \frac{1 - r_b}{1 + r_b} \quad (12)$$

where $\beta$ ranges from 0 (no algorithmic damping and maximum accuracy) to 1 (algorithmic annihilation and minimum accuracy)

### 3.4 **Increase numerical efficiency**

A coefficient matrix requires decomposition during the numerical solution process. Efficiency can be increased by selecting two terms $u_{ij}$ such that the coefficient matrix is symmetrical.

### 3.5 **Simplify expressions and scale for convenience**

Expressions are simplified with no change in accuracy by scaling.

### 3.6 **Final coefficients**

After these steps, which require extensive algebra using a symbolic computer language, we obtain the following simple coefficients: $u_{10}$=12, $u_{11}$=6, $u_{12}$=4, $u_{13}$=3-$\beta$, $u_{20}$=-2$\beta$, $u_{21}$=$u_{22}$=$u_{23}$=-(1+$\beta$).

### 3.7 **Stability**

It may be shown that with the coefficients given in Para 3.6 the algorithm is unconditionally stable for 0<=$\beta$<=1

## 4 **COMPARISON BETWEEN EXACT AND APPROXIMATE SOLUTIONS**

Exact and approximate solutions may be compared by considering the functions $R_2(\Delta t)$ and $R_3(\Delta t)$. To enable graphical comparison, the non-dimensional functions

$$G_2 = G_2(b, z, w\Delta t) = \frac{1}{2}\sqrt{4km - c^2}\, R_2(\Delta t) \quad (13)$$

$$G_3 = G_3(b, z, w\Delta t) = mR_3(\Delta t) \quad (14)$$

are defined, where $\omega^2$=k/m and $\zeta^2$=$c^2$/4km. These functions are plotted for $\zeta$=0 and $\zeta$=0.1 in Figures 1 to 4. It will be observed from these graphs that the algorithm produces accurate results for time steps up to a quarter of the period of vibration with no algorithmic damping and up to a sixth of the period of vibration with algorithmic annihilation.

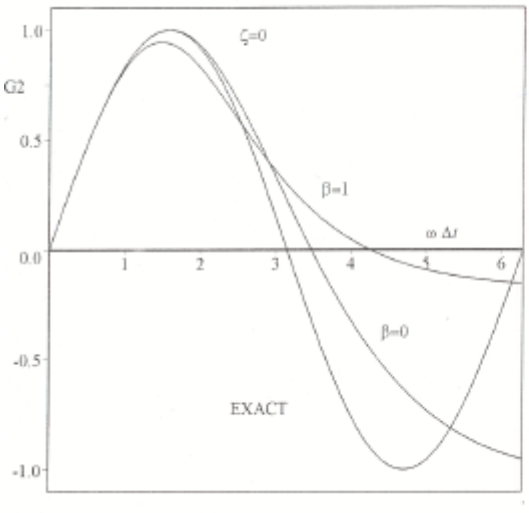


Figure 1. The function $G_2$, $\zeta$=0.0

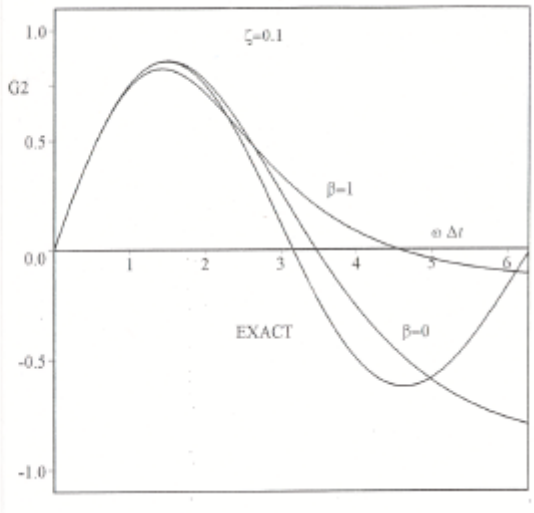


Figure 2. The function $G_2$, $\zeta$=0.1 .

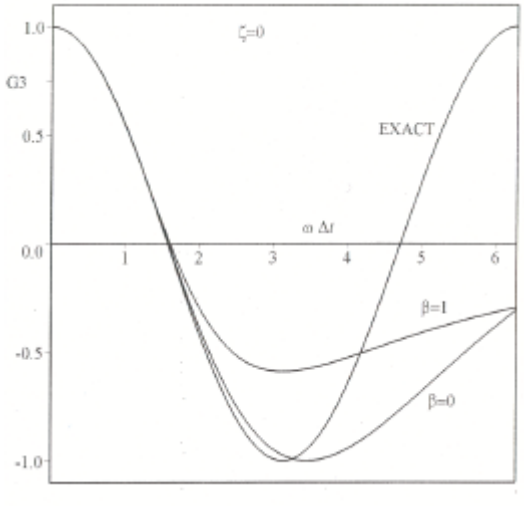


Figure 3. The function $G_3$, $\zeta$=0.0

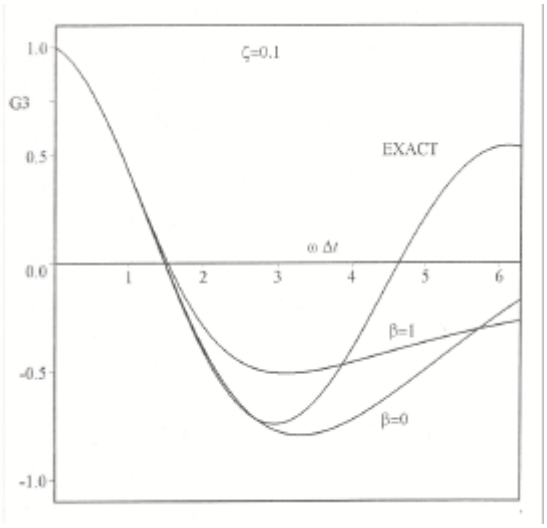


Figure 4. The function $G_3$, $\zeta$=0.1 .

## 5 **WEIGHTED SUM EQUIVALENT**

### 5.1 **General case**

With linear problems, the "areas" and "moments of area" of the weight functions are required, but not the weight functions. However, to apply the method to non-linear problems it is necessary to prescribe weight functions with the same values $u_{ij}$. It is a simple procedure to determine cubic functions with the same values $u_{ij}$, but of more use are combinations of Dirac functions at prescribed sampling times. The weight function

$$W_i(t) = w_{i1}d(t - t_{i1}\Delta t) + w_{i2}d(t - t_{i2}\Delta t), \qquad i = 1,2 \tag{15}$$

where $\delta$ is the Dirac delta function, therefore consists of concentrated weights $w_{i1}$ and $w_{i2}$ at the sampling times $\tau_{i1}\Delta t$ and $\tau_{i1}\Delta t$. We find

$$\frac{w_{11}}{\Delta t} = 6(1 - \frac{3b}{\sqrt{3+9b^2}}), \quad \frac{w_{12}}{\Delta t} = 6(1 + \frac{3b}{\sqrt{3+9b^2}}), \quad \frac{w_{21}}{\Delta t} = 1 - b, \quad \frac{w_{22}}{\Delta t} = -1 - b \qquad (16)$$

$$t_{11} = \frac{1}{2}(1-b) - \frac{1}{6}\sqrt{3+9b^2}, \quad t_{12} = \frac{1}{2}(1-b) + \frac{1}{6}\sqrt{3+9b^2}, \quad t_{21} = 0, \quad t_{22} = 1 \qquad (17)$$

These weight functions enable the method to be viewed as a weighted sum procedure. Using the weights and locations given by Equations 16 and 17, and noting $\tau_{21}$=0 and $\tau_{22}$=1 for all $\beta$, the weighted integral equations reduce to

$$w_{11} r(t_{11}\Delta t) + w_{12} r(t_{21}\Delta t) = 0 \qquad (18)$$

and

$$w_{21} r(0) + w_{22} r(1) = 0 \qquad (19)$$

where $r(\tau\Delta t)$ is the residual at time $\tau\Delta t$.

### 5.2 **Special case: no algorithmic damping**

With no algorithmic damping, ie $\beta$=0, the weights and sampling times become

$$\frac{w_{11}}{\Delta t} = 6, \quad \frac{w_{12}}{\Delta t} = 6, \quad \frac{w_{21}}{\Delta t} = 1, \quad \frac{w_{22}}{\Delta t} = -1 \qquad (20)$$

and

$$t_{11} = \frac{1}{2} - \frac{\sqrt{3}}{6}, \quad t_{12} = \frac{1}{2} - \frac{\sqrt{3}}{6}, \quad t_{21} = 0, \quad t_{22} = 1 \qquad (21)$$

The sampling times are therefore the beginning and end of the time step and two Gauss points. Thus in the linear case with no algorithmic damping, the method equates to zero the sum of the residuals at the Gauss points, and also equates the residuals at the beginning and end of the time step. If the applied force is approximated by a polynomial up to cubic order, the residual is a cubic function, and consideration of cubic functions with these characteristics reveals the residual is zero twice within the time step. The equivalent weighted sum procedure is readily applied to non-linear problems.

## 6 **APPLICATIONS**

The method has been successfully applied to the solution of the SDOF Duffing equation, and to Bathe's pendulum (Bathe and Wilson, 1973). The latter problem has been considered in detail by Crisfield and Shi (1994), as most numerical solutions lead to a phenomenon called locking. Using the proposed method, locking is avoided, although a large number of time steps are necessary for high accuracy due to the contribution of high frequencies in the solution as discussed by Golley and Petrolito (2002). The algorithm has also been used successfully by Mitchell (2001) in a detailed investigation of lead core hysteretic bearings used in seismic isolation of buildings.

## 7 **CONCLUSIONS**

A new unconditionally stable algorithm for the linear and non-linear dynamic analysis of structures has been presented. The algorithm is developed using two unspecified weighted functions, with

coefficients chosen to satisfy a number of criteria, including optimising accuracy and numerical efficiency. Comparison of the approximate algorithm with exact solutions for the linear case shows that large time steps are feasible. An equivalent weighted sum method is derived from the optimum solution, which may be used to solve MDOF problems with material and geometric nonlinearities.

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